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Fractal Dimensions and Homeomorphic Conjugacies

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We investigate the behavior of the spectrum of singularities associated with the invariant measure of some dynamical systems under nonsmooth coordinate changes. When the homeomorphic conjugacy is not Lipschitz continuous, we discuss how its singularities can affect the whole set of generalized fractal dimensions. We give applications to homeomorphisms that conjugate critical circle maps with irrational (golden mean) winding numbers. We present numerical studies corroborating the theoretical predictions.

KEY WORDS: Fractal dimensions; spectrum of singularities; invariant measures; homeomorphic conjugacies; critical circle maps.

1. INTRODUCTION

Much interest has been paid recently to the so-called generalized fractal dimensions.[1-4] First defined by Renyi,[1] these dimensions have been shown to be intimately related to the spectrum of singularities associated with a probability measure μ.[5-7] In the context of dynamical systems, μ is a measure that is invariant under the dynamics.[8] For the sake of simplicity we work in this paper with one-dimensional dynamical systems, although we hope that some of our results can be extended to higher dimensions. We assume that μ is concentrated on $[0, 1]$, i.e., $\mu([0, 1]) = 1$, where $\{I^U\}_N$ is the uniform partition of $[0, 1]$ by intervals of length $1/N$.

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Then, using the thermodynamic formalism,[9,10] one can introduce a partition function $Z_N(\beta)$, which is defined for $\beta \in R$ by

$$Z_N(\beta) = \sum_{I \in \{I^U\}_N} \mu(I)^\beta \tag{1.1}$$

Hence it is natural to define the uniform free energy $F^U(\beta)$ of the measure μ when the following limit exists:

$$F^U(\beta) = \lim_{N \to +\infty} -\log Z_N(\beta)/\log(N) \tag{1.2}$$

Formally deriving (1.1) shows that in general $F^U(\beta)$ is convex in β; then, Legendre-transforming $F^U(\beta)$, one gets the Gibbs free energy $G^U(\alpha)$,

$$G^U(\alpha) = \alpha\beta - F^U(\beta), \qquad \alpha = dF^U(\beta)/d\beta \tag{1.3}$$

The Gibbs free energy is also called the spectrum of singularities[5-7] of the measure μ; the generalized fractal dimensions are defined by

$$D_\beta = (\beta - 1)^{-1} F^U(\beta) \tag{1.4}$$

These definitions can be understood by exploring the singular behavior of the measure $\mu(I) \sim [L(I)]^\alpha$ in the limit of zero Lebesgue measure $[L(I) \to 0]$ for the interval I. Let us introduce the two scaling functions $\alpha^+(x)$ and $\alpha^-(x)$, which describe the local singularity strength[11] of μ,

$$\alpha^+(x) = \limsup_{x \in I, L(I) \to 0} \log \mu(I)/\log L(I) \tag{1.5}$$

$$\alpha^-(x) = \liminf_{x \in I, L(I) \to 0} \log \mu(I)/\log L(I) \tag{1.6}$$

Then the main question which has been addressed in Refs. 7 and 11 is: What can we say about the sets M_α^+ and M_α^- as defined by

$$M_\alpha^\pm = \{x \in I \mid \alpha^\pm(x) = \alpha\} \tag{1.7}$$

and which correspond, respectively, to sets of points where the singular behavior of μ is governed by the same exponent $\alpha \in R$? When μ is the invariant measure of a regular, locally expanding Markov map, it has been shown rigorously[11] that the Hausdorff dimensions (HD) of M_α^+ and M_α^- are equal to each other and to the Gibbs free energy,

$$\mathrm{HD}(M_\alpha^+) = \mathrm{HD}(M_\alpha^-) = G^U(\alpha) \tag{1.8}$$

Although the interpretation of $G^U(\alpha)$ remains unclear for more general μ, we shall assume that this result applies to the set of measures we shall investigate in the remainder of this paper.

As reported in Refs. 12 and 13, the generalized fractal dimensions D_β as well as the uniform Gibbs free energy $G^U(\alpha)$ are quantities that are measurable from experimental data. This raises the question of the actual invariance of these functions under some coordinate changes. In a recent paper,[14] some examples are given to illustrate that some dimensions are invariant and some are not under changes of variables that are differentiable except at a finite number of points. The aim of the present paper is to show that the whole spectrum of singularities of μ and in addition the whole set of generalized fractal dimensions may be drastically affected by nonsmooth coordinate changes. Our goal is not only to demonstrate this effect, but also to try to quantify the alteration of D_β and $G^U(\alpha)$ in terms of the singularities of the associated homeomorphic conjugacy.

This paper is organized as follows. In Section 2 we show that, under some working hypotheses, the functions $F^U(\beta)$ and $G^U(\alpha)$ are invariant under Lipschitz continuous changes of coordinates. When the change of variables does not satisfy this requirement, we discuss how the singularities of the homeomorphic conjugacy affect the spectrum of singularities of the measure μ. In Section 3 we give applications to homeomorphisms that conjugate critical circle maps with irrational winding numbers.[15] We derive relations between the spectrum of singularities of critical circle maps that belong to different universality classes[15–17] and present numerical studies confirming the theoretical predictions. We conclude in Section 4 and review some important examples of measures arising in one-dimensional dynamical systems for which our theoretical results are likely to apply. We comment about the generalization of these results to higher dimensional dynamical systems.

2. HOMEOMORPHIC CONJUGACIES OF DYNAMICAL SYSTEMS

2.1. Basic Results

Two dynamical systems $f_1\colon [0,1] \to [0,1]$ and $f_2\colon [0,1] \to [0,1]$ with respective invariant measures μ_1 and μ_2 are said to be conjugate if there exists a homeomorphism (change of coordinates) $h\colon [0,1] \to [0,1]$ such that

$$f_2 = h^{-1} \circ f_1 \circ h \tag{2.1}$$

$$\mu_2(I) = \mu_1(hI), \qquad \forall I \subset [0,1] \tag{2.2}$$

where I is any measurable subset of $[0, 1]$. By definition h is a continuous and strictly monotone function. Hence, one can associate to h the following measures:

$$\mu_h(I) = L(hI) \tag{2.3}$$

$$\mu_{h^{-1}}(I) = L(h^{-1}I) \tag{2.4}$$

where L denotes the Lebesgue measure. The scaling exponents (1.5) and (1.6) of these measures $\alpha_h^{\pm}(x)$ and $\alpha_{h^{-1}}^{\pm}(x)$ are local Holder exponents of h and h^{-1}, respectively. A straightforward calculation shows that these exponents are not independent, but satisfy the relation

$$\alpha_h^{\pm}(x) \cdot \alpha_{h^{-1}}^{\mp}(hx) = 1 \tag{2.5}$$

This identity merely means that if h scales at x with the exponent α, then h^{-1} will scale at hx with the exponent $1/\alpha$.

One of our basic hypotheses in this paper is to assume that all the measures scale exactly, i.e., the lim sup (respectively inf) can be replaced by the simple limit

$$\alpha(x) = \alpha^{+}(x) = \alpha^{-}(x) \tag{2.6}$$

and this for α_h and $\alpha_{h^{-1}}$ as well as for the scaling exponents α_1 and α_2 associated with μ_1 and μ_2, respectively. This assumption is in general too strong and it is very likely that our results still hold under much weaker conditions. Unfortunately, we have not been able to extend our proof to more general situations.

Within this hypothesis, it is easy to determine how the scaling functions $\alpha_1(x)$ and $\alpha_2(x)$ are related by the singularities of the homeomorphism h which conjugates the two dynamics. From

$$\alpha_2(x) = \lim_{x \in I, L(I) \to 0} \log \mu_2(I)/\log L(I) = \lim_{x \in I, L(I) \to 0} \log \mu_1(hI)/\log L(I)$$

one gets [upon multiplying the right-hand side of this equality by $\log L(hI)/\log L(hI) = 1$] the relation

$$\begin{aligned} \alpha_2(x) &= \lim_{x \in I, L(I) \to 0} [\log \mu_1(hI)/\log L(hI)] \cdot [\log L(hI)/\log L(I)] \\ &= \alpha_1(hx) \cdot \alpha_h(x) \end{aligned} \tag{2.7}$$

Then, using (2.5), we deduce an additional relation

$$\alpha_1(hx) = \alpha_2(x) \cdot \alpha_{h^{-1}}(hx) \tag{2.8}$$

which describes the local interaction of the singularities of the homeomorphic conjugacy with the singularities of the invariant measure.

2.2. Lipschitz Continuous Homeomorphic Conjugacies

It immediately follows from (2.7) that if the homeomorphism h is Lipschitz continuous, i.e., $\alpha_h(x) = \alpha_{h^{-1}}(x) = 1$, then the scaling functions are invariant under h,

$$\alpha_1(hx) = \alpha_2(x) \tag{2.9}$$

Therefore, the sets $M_\alpha^{(1)}$ and $M_\alpha^{(2)}$ associated via (1.5)–(1.7) with μ_1 and μ_2, respectively, are mapped under h onto each other as

$$M_\alpha^{(1)} = hM_\alpha^{(2)} \tag{2.10}$$

Since the Hausdorff dimension is invariant under a Lipschitz continuous change of variables,[18] this implies from (1.8) that the spectra of singularities of μ_1 and μ_2 are the same

$$G_1^U(\alpha) = G_2^U(\alpha) \tag{2.11}$$

Consequently, from (1.2)–(1.4), we end with the result that the uniform free energy $F^U(\beta)$ and the generalized fractal dimensions D_β are invariant under a Lipschitz continuous homeomorphic conjugacy.

2.3. Homeomorphisms that Conjugate a Dynamical System to its Inverse

Among the nonsmooth homeomorphic conjugacies, let us first consider the class of homeomorphisms that conjugate f to f^{-1}, i.e., such that $h = h^{-1}$. Then, from (2.5) one obtains $\alpha_h(hx) = 1/\alpha_h(x)$, which means that h maps onto each other the sets M_α^h and $M_{1/\alpha}^h \subset [0, 1]$,

$$M_\alpha^h = hM_{1/\alpha}^h \tag{2.12}$$

But by definition h scales at each point of $M_{1/\alpha}^h$ with an exponent $1/\alpha$; thus, it follows that the Hausdorff dimension of this set in general is multiplied by α. This results in the relation

$$G_h(\alpha) = \alpha G_h(1/\alpha) \tag{2.13}$$

which Legendre-transforms into

$$F_h \circ F_h = id \tag{2.14}$$

Because the invariant measure of a dynamical system is not only invariant under f but also under f^{-1}, the relations (2.13) and (2.14) are of main interest in the context of this paper.

2.4. Homeomorphisms that Conjugate Some Dynamical Systems to Ones with Nonsingular Invariant Measure

In general the singularities of h will interact with those of the measure μ, yielding nontrivial behavior of the Gibbs free energy under h. Of particular interest are the homeomorphisms that conjugate a dynamical system f_1 to a dynamical system f_2 with nonsingular invariant measure, i.e., $\alpha_2(x)=1$. Then from (2.8) one gets

$$\alpha_1(x)=\alpha_{h^{-1}}(x) \tag{2.15}$$

This relation tells us that the sets $M_\alpha^{(1)}$ and $M_\alpha^{h^{-1}} \subset [0, 1]$, where μ_1 and $\mu_{h^{-1}}$ behave, respectively, with the same scaling exponent α, are identical; consequently, they have the same Hausdorff dimension

$$G_1^U(\alpha)=G_{h^{-1}}(\alpha) \tag{2.16}$$

Now from (2.5) one deduces easily that

$$M_\alpha^{h^{-1}}=hM_{1/\alpha}^h \tag{2.17}$$

Since by definition h scales at every point of $M_{1/\alpha}^h$ with an exponent $1/\alpha$, the Hausdorff dimension in general is multiplied by α and (2.16) becomes

$$G_1^U(\alpha)=G_{h^{-1}}(\alpha)=\alpha G_h(1/\alpha) \tag{2.18}$$

Therefore, from the relation (2.16) between the spectrum of singularities of μ_1 and that of the homeomorphic conjugacy, it is easy to conclude that the corresponding free energies satisfy

$$F_1^U(\beta)=F_{h^{-1}}(\beta) \tag{2.19}$$

that is, by Eq. (1.3), the generalized fractal dimensions D_β associated with μ_1 are the same as those associated with $\mu_{h^{-1}}$. Moreover, the well-known properties of the Legendre transform allow us to transform (2.18) into the following identity:

$$F_{h^{-1}} \circ F_h = id \tag{2.20}$$

2.5. Relation between the Uniform Free Energy and the Dynamical Free Energy

Instead of taking the equipartition $\{I^U\}_N$ to define the uniform free energy $F^U(\beta)$ in (1.2), it may be more convenient, as already suggested by several numerical studies,[19] to work with the dynamical partition as defined by $\mu^D(I)=1/N$, $\forall I \in \{I^D\}_N$, and to consider the dynamical free energy:

$$F^D(\beta)=\lim_{N\to+\infty} -\log\left\{\sum_{I\in\{I^D\}_N}[L(I)]^\beta\right\}\Big/\log(N) \tag{2.21}$$

We wish to establish a relation between F^U and F^D. Let h be the homeomorphism that conjugates a dynamics f with invariant measure μ_f to a dynamics having L as invariant measure. As is easily seen from (2.2), we have $\mu_f(hI^U)=L(I^U)=1/N$. This shows that $I^D=hI^U$, and from $L(I^D)=L(hI^U)=\mu_h(I^U)$ one gets

$$F^D(\beta)=F_h(\beta) \tag{2.22}$$

Now from (2.4) one can also write $\mu_f(I^U)=L(h^{-1}I^D)=\mu_{h^{-1}}(I^D)$, which leads to

$$F^U(\beta)=F_{h^{-1}}(\beta) \tag{2.23}$$

Then from the identity (2.20) one deduces easily that the dynamical free energy is the inverse function of the uniform free energy:

$$F^D\circ F^U=id \tag{2.24}$$

This result was first discovered in Ref. 11, but with a different proof and under different hypotheses. It follows from the Legendre transform properties that the spectrum of singularities computed with the dynamical partition can be derived from the spectrum of singularities computed with the equipartition (and vice versa) according to the relation [see (2.18)]

$$G^D(\alpha)=\alpha G^U(1/\alpha) \tag{2.25}$$

3. FRACTAL DIMENSIONS OF THE GOLDEN MEAN TRAJECTORIES AT THE ONSET OF CHAOS

3.1. Circle Map Models for the Transition from Quasiperiodicity to Chaos

In the past few years there has been much interest in the study of the universal properties of the transition to chaos.[20-23] Among the well-known scenarios, special attention has been paid to the transition from quasiperiodicity with two incommensurate frequencies to "weak turbulence." The usual approach consists in modeling the Poincaré surface section with invertible analytic maps of the annulus.[24–26] In the limit of infinite area contraction, these maps reduce to analytic maps of the circle. A prototype of such maps is the two-parameter sine family,[27]

$$\theta_{n+1} = f^{(S)}_{K,\Omega}(\theta_n) = \theta_n + \Omega - K/2\pi \sin(2\pi\theta_n) \tag{3.1}$$

where the parameter K provides the strength of the nonlinearities and the parameter Ω sets the rate of rotation. Let $W^* = (\sqrt{5}-1)/2$ be the golden mean. Then for every $K<1$, there exists a $\Omega^*(K)$ such that the winding number

$$W(K, \Omega) = \lim_{n \to +\infty} [f^{(S)n}_{K,\Omega}(\theta) - \theta]/n \tag{3.2}$$

is strictly equal to $W(K, \Omega^*(K)) = W^*$. The mapping (3.1) is the lift of a diffeomorphism of the circle, i.e., $f^{(S)}_{K,\Omega}$ mod 1 is a diffeomorphism of the circle. Since W^* belongs to the set of winding numbers defined by Herman,[28] $f^{(S)}_{K,\Omega^*}$ is analytically conjugated to a pure rotation $f^{(R)}_{W^*}$,

$$\theta_{n+1} = f^{(R)}_{W^*}(\theta_n) = \theta_n + W^* \tag{3.3}$$

where the shift (which is also the winding number of the mapping) is equal to the golden mean W^*. Since, from the previous section, the spectrum of singularities is invariant under a smooth change of variables, the spectrum of singularities of the golden mean trajectory in the sine map is the same as for a pure rotation, i.e., it is trivial[29] with a single index $\alpha = 1$. This implies that the corresponding generalized fractal dimensions are identically $D_\beta = 1$.

At $K=1$, the sine map fails to be a diffeomorphism, i.e., $f^{(S)-1}_{K,\Omega^*}$ is not differentiable everywhere, because of the cubic inflection point. *From now on we shall write* $f^{(S)}_{\Omega}$ *instead of* $f^{(S)}_{K=1,\Omega}$. This critical line is of physical interest, since it marks the onset of chaos for quasiperiodic trajectories.[15–17] Shenker[15] was the first one to discover how the universal properties of this scenario are related to the nature of the inflection point of the critical circle map. He found that the distances around $\theta = 0$ scale down by a universal factor $\alpha_{\mathrm{gm}} = 1.2885...$, when the superstable golden-mean trajectory is truncated at two consecutive Fibonacci numbers ($F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geqslant 2$),

$$\alpha_{\mathrm{gm}} = \lim_{n \to +\infty} |[\{f^{(S)}_{\Omega^*}\}^{F_{n-1}}(0) - F_n]/[\{f^{(S)}_{\Omega^*}\}^{F_n}(0) - F_{n+1}]| \tag{3.4}$$

The distance between the origin and its nearest neighbor is in fact the largest distance separating two neighboring points of the superstable orbit (zero belongs to the orbit) with winding number $W_n = F_n/F_{n+1}$ as computed with $f_{\Omega}^{(S)}$ for $\Omega = \Omega_n$. (The W_n are the Farey approximants of the golden mean.) It corresponds to the most rarefied region on the circle and thus will be the leading contribution in the dynamical free energy (2.21) in the limit $\beta \to +\infty$. Denote by μ_{Ω} the invariant measure of $f_{\Omega}^{(S)}$. Then

$$\mu_{\Omega_n}([0, \{f_{\Omega_n}^{(S)}\}^{F_n}(0)]) = 1/F_{n+1} \sim W^{*n}$$

where $\{f_{\Omega_n}^{(S)}\}^{F_n}(0)$ is the nearest neighbor to the origin. From Shenker's results, one knows that

$$L([0, \{f_{\Omega_n}^{(S)}\}^{F_n}(0)]) \sim \alpha_{\mathrm{gm}}^{-n}$$

Then one gets

$$\begin{aligned}\lim_{\beta \to +\infty} F^D(\beta) &\sim -\beta \lim_{n \to +\infty} (\log \alpha_{\mathrm{gm}}^{-n})/\log F_{n+1} \\ &\sim -\beta \log \alpha_{\mathrm{gm}}/\log W^* \end{aligned} \tag{3.5}$$

Then, via the Legendre transformation, one obtains a lower bound for the scaling exponent of the invariant measure as computed with the dynamical partition $G^D(\alpha)$:

$$\alpha_{\min} = -\log \alpha_{\mathrm{gm}}/\log W^* \tag{3.6}$$

From the cubic nature of the inflection point, the most rarefied region of the trajectory is mapped onto the most concentrated region,

$$\lim_{x=0 \in I, L(I) \to 0} \log L(f_{\Omega^*}^{(S)}(I))/\log L(I) = 3 \tag{3.7}$$

This region will be the leading contribution in the dynamical free energy in the limit $\beta \to -\infty$,

$$\lim_{\beta \to -\infty} F^D(\beta) \sim -3\beta \log \alpha_{\mathrm{gm}}/\log W^* \tag{3.8}$$

from which one can deduce an upper bound for the scaling exponent α,

$$\alpha_{\max} = -3 \log \alpha_{\mathrm{gm}}/\log W^* \tag{3.9}$$

Now if one make use of the relations (2.18), (2.20), and (2.24) between the dynamical and uniform Gibbs free energies, (3.6) and (3.9) lead to the following universal prediction for the range of scaling exponent as computed with the uniform partition $G^U(\alpha)$,[7]

$$\begin{aligned}\alpha &\in [\alpha(f_{\Omega^*}^{(S)}(0)), \alpha(0)] \\ &\in [-\tfrac{1}{3}\log W^*/\log \alpha_{\mathrm{gm}}, -\log W^*/\log \alpha_{\mathrm{gm}}] \end{aligned} \tag{3.10}$$

At this point let us mention that in the remainder of this section we will work with the uniform partition, but for the sake of simplicity in the notation we will omit the superscript U.

In light of this theoretical result, it is important to note that in Ref. 7, numerical simulations of different cubic maps strongly suggest that not only the range of scaling exponent, but the whole spectrum of singularities of the golden mean orbits is universal at the onset of chaos. Recent measurements in a periodically forced Rayleigh–Benard experiment have confirmed the theoretical speculations.[12]

To check for universality, it is important to investigate critical circle maps that differ in the order of the inflection point. In this section we mainly consider the one-parameter family

$$f_\Omega^{(z)}(\theta) = \begin{cases} \Omega + (1/2)^{1-z}\,\theta^z \bmod 1, & \theta < 1/2 \\ \Omega + 1 - (1/2)^{1-z}\,(1-\theta)^z \bmod 1, & \theta \geqslant 1/2 \end{cases} \tag{3.11}$$

As for the critical sine map (3.1), the inflection point of $f_\Omega^{(z)}$ is $\theta = 0$, but its order is z. We shall denote by $h_{z,z'}$ the homeomorphism that conjugates $f_{\Omega*}^{(z)}$ to $f_{\Omega*}^{(z')}$,

$$f_{\Omega*}^{(z')} = h_{z,z'}^{-1} \circ f_{\Omega*}^{(z)} \circ h_{z,z'} \tag{3.12}$$

The thermodynamic quantities of the measure $\mu_{z,z'}$ associated with $h_{z,z'}$ will be the free energy $F_{z,z'}(\beta)$ and the Gibbs free energy $G_{z,z'}(\alpha)$. We shall replace z or z' by S or R when will refer to the sine map (3.1) or the shift map (3.3).

Numerical Algorithm to Compute $F_{z,z'}(\beta)$ and $G_{z,z'}(\alpha)$. The algorithm we use to estimate numerically $F_{z,z'}(\beta)$ and $G_{z,z'}(\alpha)$ consists in approaching the superstable golden mean trajectories of $f_{\Omega*}^{(z)}$ and $f_{\Omega*}^{(z')}$ by successive Farey truncations $W_n = F_n/F_{n+1}$ to the continued fraction expansion of W^*. We determine $\Omega_n^{(z)}$ and $\Omega_n^{(z')}$ such that $f_{\Omega_n}^{(z)}$ and $f_{\Omega_n}^{(z')}$ display a W_n-cycle containing the origin. Then we construct the partition function

$$\Gamma^n(\beta, F_{z,z'}) = \sum_{i=1}^{F_{n+1}} [l_i^{(z)}]^\beta / [l_i^{(z')}]^{F_{z,z'}} \tag{3.13}$$

where $l_i^{(z)} = \theta_{i+F_n}^{(z)} - \theta_i^{(z)}$ are the distances between next neighboring points in the trajectory generated by $f_{\Omega_n}^{(z)}$. Then, assuming that the measure $\mu_{z,z'}$ possesses an exact recursive structure, one deduces the free energy $F_{z,z'}$ from the requirement that

$$\Gamma^n(\beta, F_{z,z'}) = 1 \qquad (n \to +\infty) \tag{3.14}$$

To improve the convergence of the free energy curve $F_{z,z'}(\beta)$ as $W_n \to W^*$, we make use of the ratio trick usually employed in similar computations[7,29]

$$\Gamma^n(\beta, F_{z,z'})/\Gamma^{n-1}(\beta, F_{z,z'}) = 1 \tag{3.15}$$

Then we Legendre transform $F_{z,z'}(\beta)$ to extract the Gibbs free energy $G_{z,z'}(\alpha)$ [see (1.3)].

3.2. Homeomorphisms that Conjugate Critical Circle Maps to a Pure Rotation

As discussed in Ref. 17, the (incremental) homeomorphism that conjugates a (cubic) critical circle map to a diffeomorphism with the same winding number, if it exists, is nowhere differentiable. In particular, this is true for the homeomorphisms that conjugate any of the critical maps $f^{(z)}_{\Omega^*}$ with golden mean winding number $W = W^*$ to a pure rotation of angle W^*. Figure 1a illustrates the homeomorphism $h_{\mathrm{S,R}}$ that conjugates the critical sine map (3.1) $f^{(\mathrm{S})}_{\Omega^*}$ to the shift map (3.3) $f^{(\mathrm{R})}_{W^*}$. Since $f^{(\mathrm{S})}_{\Omega^*}$ has a critical point at the origin, $h_{\mathrm{S,R}}$ must have[17] infinite derivative at zero and/or zero derivative at W^*. As shown in Fig. 1a, when enlarging some part of the homeomorphic conjugacy one can show that both singularities are present at $n_1 W^* - n_2$ with $n_1 < 0$ as created by the action of f^{-1} and with $n_1 > 0$ as created by f.

According to Eq. (2.19), the free energy $F_{\mathrm{S}}(\beta)$ should be equal to $F_{h^{-1}_{\mathrm{S,R}}}(\beta)$, which can be denoted as $F_{\mathrm{R,S}}(\beta)$ from (3.13) and (3.15). In Fig. 1b, we present the spectrum of singularities associated with μ_{S}, which from (2.16) is equal to

$$G_{\mathrm{S}}(\alpha) = G_{\mathrm{R,S}}(\alpha) \tag{3.16}$$

This spectrum is in remarkable agreement with the "$f(\alpha)$" spectrum computed directly from the sine map in Ref. 7. This can be understood very easily since, from (3.13),

$$\begin{aligned}\Gamma^n(\beta, F_{\mathrm{R,S}}) &= \sum_{i=1}^{F_{n+1}} [l_i^{(\mathrm{R})}]^{\beta}/[l_i^{(\mathrm{S})}]^{F_{\mathrm{R,S}}} \\ &= \frac{1}{[F_{n+1}]^{\beta}} \sum_{i=1}^{F_{n+1}} [l_i^{(\mathrm{S})}]^{-F_{\mathrm{R,S}}}\end{aligned} \tag{3.17}$$

where we have used the fact that the trajectory points of the rotation operation are uniformly distributed on the circle. Equation (3.17) is the partition function from which Halsey *et al.*[7] defined and measured what they named the generalized fractal dimensions D_q and the spectrum of singularities $f(\alpha)$ and which have to be identified as $D_{\mathrm{R,S}}(\beta) = (\beta - 1)^{-1} F_{\mathrm{R,S}}(\beta)$ and $G_{\mathrm{R,S}}(\alpha)$, respectively.

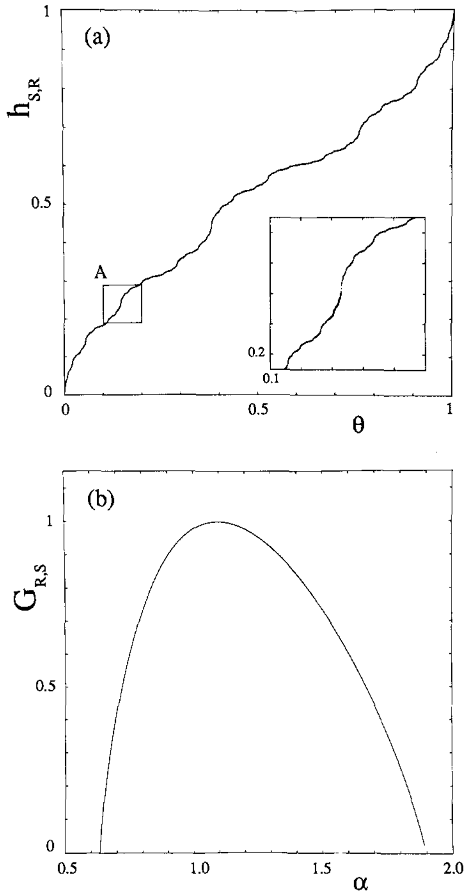

Fig. 1. (a) The homeomorphism $h_{S,R}$, which conjugates the critical sine map (3.1) with $K=1$ and $\Omega=\Omega^*$ with the shift map (3.3) with winding number $W=W^*$. The insert corresponds to an enlargement of the square A. (b) The corresponding Gibbs free energy $G_{R,S}(\alpha)=G_{h_{S,R}^{-1}}(\alpha)$ as computed via (3.13) and (3.15).

3.3. Homeomorphisms that Conjugate Two Critical Circle Maps in the Same Class of Universality

It was conjectured in Ref. 17 that the universal properties of the critical golden mean trajectories reflect the fact that the homeomorphism that conjugates two circle maps with the same inflection point is a once continuously differentiable diffeomorphism. In Fig. 2a we compute the homeomorphism $h_{S,3}$ that conjugates the critical sine map (3.1) $f^{(S)}_{\Omega^*}$ to $f^{(3)}_{\Omega^*}$

defined in (3.11). In contrast to $h_{S,R}$, there is no evidence of any singularity in $h_{S,3}$ and this can be seen at any scale when enlarging any part of this function. This smoothness results in a trivial Gibbs free energy $G_{S,3}(\alpha = 1) = 1$, as can be checked in Fig. 2b, where we compute $G_{S,3}(\alpha)$ for different Farey approximants W_n of the golden mean using (3.13) and (3.15). When we proceed in the Farey sequence, the range of α values reduces considerably and the whole $G_{S,3}(\alpha)$ curve shrinks dramatically as n increases. Since we have performed our computation by fixing arbitrarily $\beta_{max} = -\beta_{max} = 150$, we notice that $G_{S,3}(\alpha)$ not only shrinks, but its end

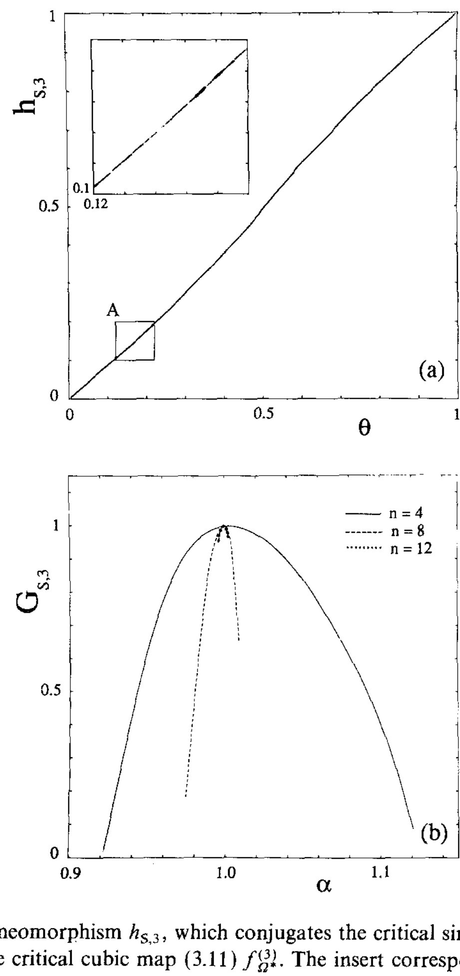

Fig. 2. (a) The homeomorphism $h_{S,3}$, which conjugates the critical sine map (3.1) with $K = 1$ and $\Omega = \Omega^*$ with the critical cubic map (3.11) $f^{(3)}_{\Omega^*}$. The insert corresponds to an enlargement of the square A. (b) The corresponding Gibbs free energy $G_{S,3}(\alpha)$ as computed via (3.13) and (3.15) with different Farey approximants $W_n = F_n/F_{n+1}$ of the golden mean.

points (corresponding to these extremal values of β) take off from zero and converge to 1, which strongly suggests that $G_{S,3}(\alpha)$ actually reduces to one point in the asymptotic limit $n \to +\infty$. Hence, for $n = 12$ ($W_{12} = 144/233$), the scaling is almost trivial with $D_\beta = (\beta - 1)^{-1} F_{S,3}(\beta) \cong 1$ for values of $|\beta|$ up to ~ 100.

Our numerical experiments clearly indicate that this result is very likely to extend to any universality class and to any irrational winding number with a periodic continued fraction expansion. Therefore, given two critical quasiperiodic trajectories, the computation of the spectrum of singularities of the associated homeomorphic conjugacy can be used as a test to decide whether the underlying dynamical system belongs to the same universality class. The condition $G_{z,z'}(\alpha) = \delta_{\alpha,1}$ is a sufficient but not a necessary condition for $G_z(\alpha)$ and $G_{z'}(\alpha)$ to be identical, as we will illustrate in Section 3.5.

3.4. Homeomorphisms that Conjugate Two Critical Circle Maps in Different Classes of Universality

In Fig. 3 we show the homeomorphism $h_{S,1/2}$, which conjugates the critical sine map (3.1) $f^{(S)}_{\Omega^*}$ to the critical map (3.11) $f^{(1/2)}_{\Omega^*}$ with an inflection point of order $z = 1/2$. This homeomorphism is singular, as the computation of the Gibbs free energy $G_{S,1/2}$ in Fig. 3b strongly suggests. Obviously the scaling is nontrivial, and the range of α values characterizing the singularity of the associated measure $\mu_{S,1/2}$ is finite, with $\alpha_{max} \cong 6\alpha_{min}$.

This numerical finding can be understood theoretically, since the support $[\alpha_{min}, \alpha_{max}]$ of $G_{z,z'}(\alpha)$ can be calculated exactly. Because the inflection point of $f^{(z)}_{\Omega^*}$ is mapped onto that of $f^{(z')}_{\Omega^*}$ under $h_{z,z'}$, then $h_{z,z'}(0) = 0$. Moreover, from (2.7) we know that $h_{z,z'}$ scales at zero with the exponent

$$\alpha_{z,z'}(0) = \alpha_{z'}(0)/\alpha_z(0) \tag{3.18}$$

where $\alpha_z(0)$ and $\alpha_{z'}(0)$ are the scaling exponents at the origin associated with the measures μ_z and $\mu_{z'}$ of $f^{(z)}_{\Omega^*}$ and $f^{(z')}_{\Omega^*}$, respectively. Now, by definition from Eq. (2.1) we know that

$$h_{z,z'}[f^{(z')}_{\Omega^*}(0)] = f^{(z)}_{\Omega^*}[h_{z,z'}(0)] = f^{(z)}_{\Omega^*}(0)$$

and consequently $h_{z,z'}$ will scale around $f^{(z)}_{\Omega^*}(0)$ with the scaling exponent

$$\alpha_{z,z'}(f^{(z)}_{\Omega^*}(0)) = z\alpha_{z'}(0)/z'\alpha_z(0) \tag{3.19}$$

Equations (3.18) and (3.19) define the smallest and largest exponents of $\mu_{z,z'}$. As discussed in Section 3.5, $\alpha_z(0)$ [like $\alpha_3(0)$ in Section 3.1] can be expressed as a function of Shenker's constant $\alpha_{\rm gm}(z)$ [see (3.10)],

$$\alpha_z(0) = -\log W^*/\log \alpha_{\rm gm}(z) \tag{3.20}$$

and we end with the following prediction for the support of $G_{z,z'}(\alpha)$:

$$\begin{aligned}\operatorname{supp} G_{z,z'}(\alpha) &= [\min, \max]\{\alpha_{z'}(0)/\alpha_z(0),\ z\alpha_{z'}(0)/z'\alpha_z(0)\}\\ &= [\min, \max]\{\log \alpha_{\rm gm}(z)/\log \alpha_{\rm gm}(z'),\ z \log \alpha_{\rm gm}(z)/z' \log \alpha_{\rm gm}(z')\}\end{aligned} \tag{3.21}$$

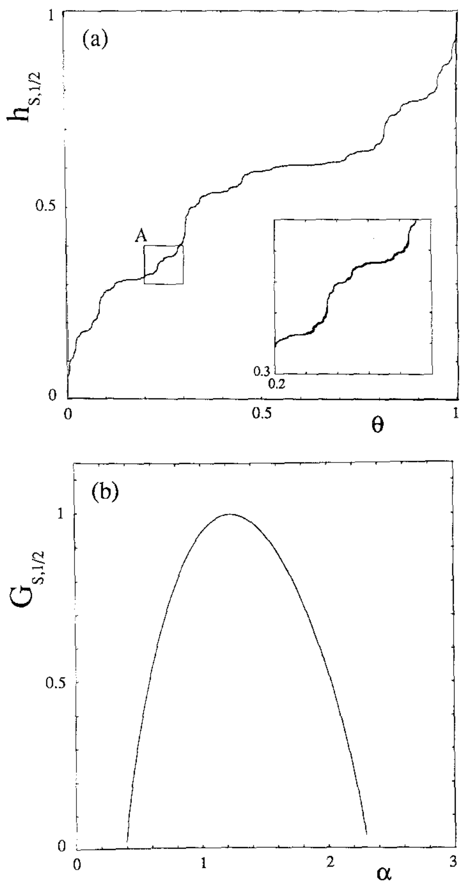

Fig. 3. (a) The homeomorphism $h_{S,1/2}$, which conjugates the critical sine map (3.1) with $K = 1$ and $\Omega = \Omega^*$ with the critical map (3.11) $f^{(1/2)}_{\Omega^*}$. The insert corresponds to an enlargement of the square A. (b) The corresponding Gibbs free energy $G_{S,1/2}(\alpha)$ as computed via (3.13) and (3.15).

From the universality hypothesis and the computation of $\alpha_{\rm gm}(z)$ (see Fig. 6a), one gets

$$\operatorname{supp} G_{S,1/2}(\alpha) = [0.3864..., 2.318...] \tag{3.22}$$

which is in complete agreement with our numerical simulations in Fig. 3b.

Unfortunately, there exists no equivalent prediction for the spectrum of singularities $G_{z,z'}(\alpha)$. From our general study in Section 2, we can nevertheless derive a relation between $G_{z,z'}$ and $G_{1/z',1/z}$. The map $f_{\Omega^*}^{(z)-1}$

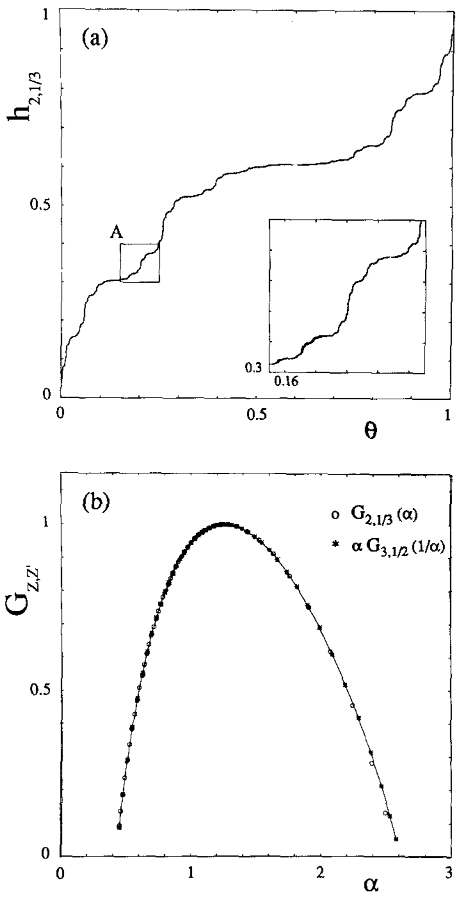

Fig. 4. (a) The homeomorphism $h_{2,1/3}$, which conjugates the critical map [see (3.11)] $f_{\Omega^*}^{(2)}$ to $f_{\Omega^*}^{(1/3)}$. The insert corresponds to an enlargement of the square A. (b) (○) The corresponding Gibbs free energy $G_{2,1/3}(\alpha)$ compared to (∗) $\alpha G_{3,1/2}(1/\alpha)$; these curves are identical, as expected from (3.24).

belongs to the universality class $1/z$, but its rotation number is $-W^*$. However, let $\mathfrak{J}$ be the map of the circle $\mathfrak{J}: \theta \to 1-\theta$. Then

$$f_{\Omega^*}^{(1/z)} = \mathfrak{J}^{-1} \circ f_{W^*}^{(\mathrm{R})} \circ f_{\Omega^*}^{(z)-1} \circ f_{W^*}^{(\mathrm{R})-1} \circ \mathfrak{J} \tag{3.23}$$

Since $\mathfrak{J}$ and the rotation $f_{W^*}^{(\mathrm{R})}$ are smooth, then $f_{\Omega^*}^{(1/z)}$ and $f_{\Omega^*}^{(z)-1}$ will have the same spectrum of singularities. Now $f_{\Omega^*}^{(z')-1}$ and $f_{\Omega^*}^{(z)-1}$ are conjugated via the homeomorphism $h_{1/z',1/z}$, which is also $h_{z,z'}^{-1}$. From (2.18) one obtains

$$G_{1/z',1/z}(\alpha) = \alpha G_{z,z'}(1/\alpha) \tag{3.24}$$

Figure 4 illustrates this result for the particular choice $z=2$, $z'=1/3$; different choices for z and z' lead to the same finding.

3.5. Homeomorphisms that Conjugate Two Critical Circle Maps in the Universality Classes z and $1/z$

As illustrated in Fig. 5, if one considers the special case $h_{z,1/z}$, then (3.24) yields the following functional equation for $G_{z,1/z}$:

$$G_{z,1/z}(\alpha) = \alpha G_{z,1/z}(1/\alpha) = G_{1/z,z}(\alpha) \tag{3.25}$$

Besides a direct comparison of $G_{z,1/z}(\alpha)$ and $G_{1/z,z}(\alpha)$, this relation can be verified by taking the logarithm of (3.25) and checking that $Y(\alpha) = \log[G_{z,1/z}(\alpha)] - \frac{1}{2}\log(\alpha)$ is a symmetric function of $\log(\alpha)$. As shown in Fig. 5b, numerical simulations corroborate this theoretical result.

Now if one comes back to the measure μ_z associated with the golden mean trajectory generated by a critical circle map with inflection point of order z, then a straightforward generalization of (3.10) leads to the following prediction for the support of $G_z(\alpha)$:

$$\alpha \in [\alpha_{\min}(z), \alpha_{\max}(z)] \tag{3.26}$$

with

$$\alpha_{\min(\max)}(z) = \min(\max)\{\alpha_z(0), \alpha_z(0)/z\}$$

$\alpha_z(0)$ is related to Shenker's scaling factor $\alpha_{\mathrm{gm}}(z)$ by the relation (3.20). Figure 6 shows the z dependence of $\alpha_{\mathrm{gm}}(z)$ as computed with the critical circle maps (3.11).

Because $f_{\Omega^*}^{(z)-1}$ belongs to the universality class $1/z$ but has a rotation number $-W^*$, one can reproduce the argument developed in (3.23) to derive, using (2.5), the relation

$$\alpha_z(0) = z\alpha_{1/z}(0) \tag{3.27}$$

i.e., from (3.26) that

$$\operatorname{supp} G_z(\alpha) = \operatorname{supp} G_{1/z}(\alpha) \tag{3.28}$$

This result is illustrated in Fig. 6b, where $[\alpha_{\min}(z), \alpha_{\max}(z)]$ is plotted versus $\log(z)$ and shown to be symmetric with respect to the pure rotation case $z = 1$, for which it is zero.

Because μ_z is not only invariant under $f_{\Omega*}^{(z)}$, but also under $f_{\Omega*}^{(z)-1}$, then, not only the support, but the whole $G_z(\alpha)$ curve is identical to $G_{1/z}(\alpha)$,

$$G_z(\alpha) = G_{1/z}(\alpha) \tag{3.29}$$

as numerically checked in Fig. 7a.

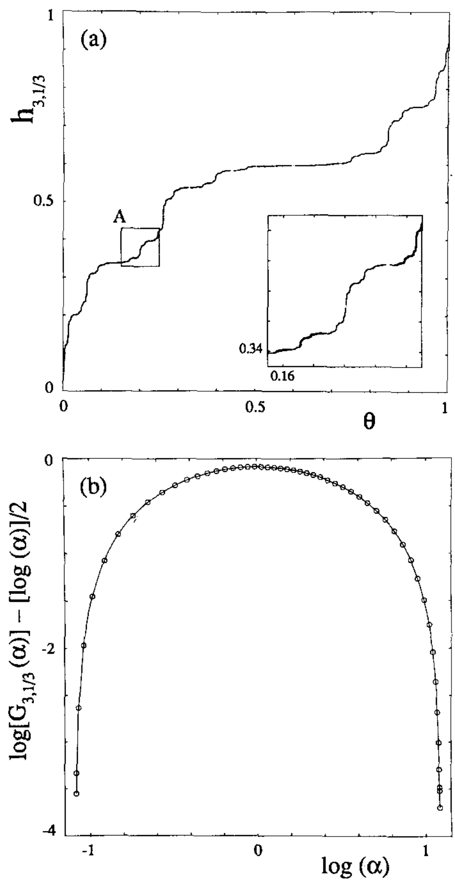

Fig. 5. (a) The homeomorphism $h_{3,1/3}$, which conjugates the critical map [see (3.11)] $f_{\Omega*}^{(3)}$ to $f_{\Omega*}^{(1/3)}$. The insert corresponds to an enlargement of the square A. (b) $Y(\alpha) = \log[G_{3,1/3}(\alpha)] - \log(\alpha)/2$ plotted versus $\log(\alpha)$. The function $Y(\alpha)$ is a symmetric function, in agreement with (3.25).

These results were anticipated in Section 2 when investigating the homorphisms that conjugate a dynamical system to its inverse and thus satisfy $h = h^{-1}$. In particular, (3.25) is nothing other than an illustration of (2.13). Equation (3.29) is an example of two dynamical systems whose invariant measures display the same nontrivial spectrum of singularities, but which nevertheless are conjugated by a nonsmooth homeomorphism. This result is supported by the numerical simulations in Fig. 7, where the spectrum of singularities and the generalized fractal dimensions for the

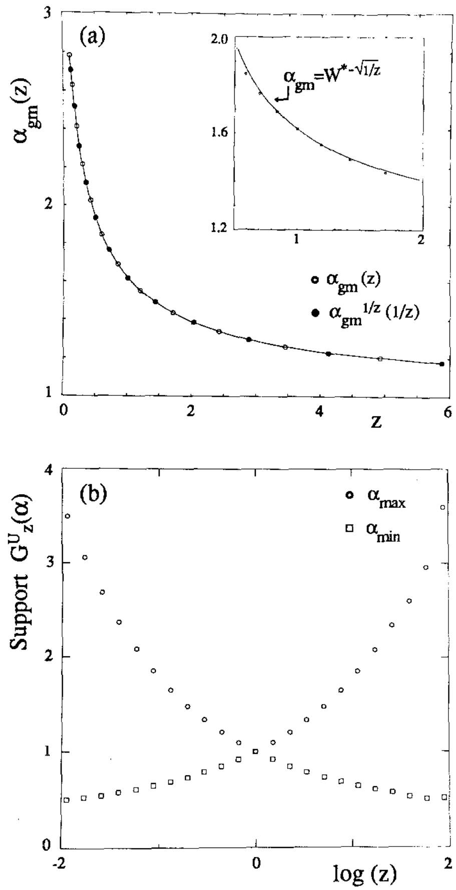

Fig. 6. (a) The z dependence of Shenker's scaling factor, (3.4), $\alpha_{gm}(z)$ as computed with the critical circle maps (3.11): (○) $\alpha_{gm}(z)$, (●) $\alpha_{gm}^{1/z}(1/z)$. The fact that the circles and the dots end on the same curve provides numerical evidence that $\alpha_{gm}(z)$ satisfies the relation (3.32). The insert corresponds to an enlargement of $\alpha_{gm}(z)$ in the neighborhood of $z=1$. (—) A fit of $\alpha_{gm}(z)$ with the approximation (3.31). (b) The support $[\alpha_{min}(z), \alpha_{max}(z)]$ of $G_z^U(\alpha)$ versus $\log(z)$. Note the symmetry with respect to $z=1$, which confirms the prediction (3.28).

universality classes 1/5, 1/3, 1/2, 2, 3, 5 are presented. Although the discrepency observed in $D_z(\beta)$ for $z = 2, 3$, and 5 is at the heart of the universality hypothesis, let us emphasize that in the spirit of this paper, it can also be seen as an illustration of the sensitivity of the spectrum of fractal dimensions under nonsmooth changes of variables.

On the $a_{gm}(z)$ Scaling Factor. From (3.20) and (3.27), it immediately follows that

$$\log \alpha_{gm}(z) = (1/z) \log \alpha_{gm}(1/z) \tag{3.30}$$

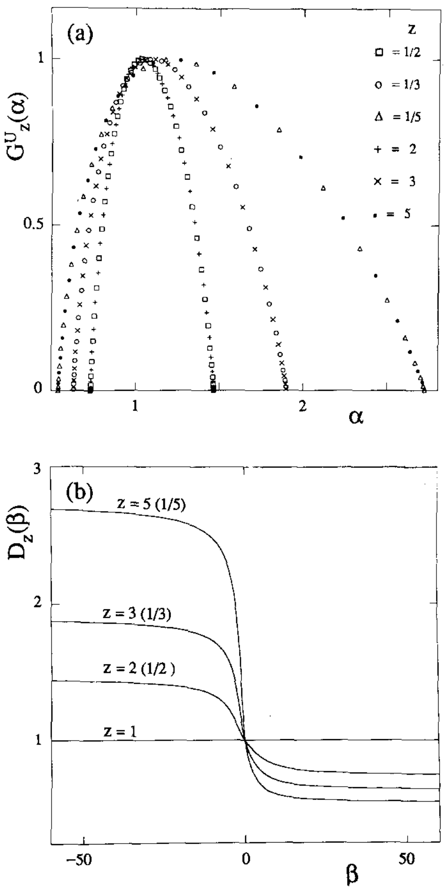

Fig. 7. (a) The spectrum of singularities $G_z^U(\alpha)$ versus α as computed with the critical circle maps (3.11) for $z = (\square)$ 1/2, ($\circ$) 1/3, ($\triangle$) 1/5, (+) 2, ($\times$) 3, and ($\bullet$) 5. Note that these numerical results strongly suggest that $G_z^U(\alpha) = G_{1/z}^U(\alpha)$. (b) The spectrum of fractal dimensions $D_z(\beta)$ versus β for $z = 1/2$, 1/3, 1/5, 1, 2, 3, and 5.

Substituting $z=e^{\xi}$ in (3.30), we obtain that the function $g(\xi)=\log[\log\alpha_{gm}(\xi)]+\xi/2$ is a symmetric function of ξ. Because $\alpha_{gm}(1)=W^{*-1}$, then one can write in a first approximation $g(\xi)=\log[-\log W^*]+O(\xi^2)$. Going back to the z variable, this leads to the following expression for Shenker's scaling factor in the neighborhood of $z=1$:

$$\alpha_{gm}(z)\cong(W^*)^{-\sqrt{(1/z)}} \tag{3.31}$$

The insert in Fig. 6a shows that this approximation is quite reasonnable for $z\in[0.7, 1.3]$. More generally, Fig. 6a also corroborates the relation (3.30), which can be rewritten as

$$\alpha_{gm}(1/z)=[\alpha_{gm}(z)]^z \tag{3.32}$$

4. DISCUSSION

In the preceding section, we examined the implications of the formalism developed in Section 2 for the study of the universal properties of the quasiperiodic trajectories at the onset of chaos. A similar approach can be used to investigate other examples of invariant measures associated with one-dimensional dynamical systems. Among the well-known Cantor sets let us briefly discuss (1) the 2^∞-cycle at the accumulation point of the period-doubling cascade, and (2) the set of irrational winding numbers at the onset of chaos.

4.1. The 2^∞-Cycle at the Accumulation Point of the Period-Doubling Cascade

The discovery of the universality of period doublings in one-dimensional discrete systems[30–32] is the origin of the analogy between the transitions to chaos observed in dissipative systems[20–22] and second-order phase transitions in equilibrium systems.[33] It is therefore not surprising that the main tool employed in critical phenomena, i.e., the renormalization group,[34] has been successfully used to understand the universal properties of the scenarios to chaos. As far as the period-doubling cascade is concerned, the existence of a fixed point for the renormalization operation[23] follows from the observation that the adherence of the asymptotic 2^∞-orbit at the accumulation point of the cascade (of almost all initial condition in the unit interval) is a Cantor set. After early attempts[35,36] to measure the Hausdorff dimension $D_0=0.537...$ of this Cantor set, the whole spectrum of fractal dimensions D_β together with the corresponding spectrum of singularities $G^U_{z=2}(\alpha)$ was calculated in Ref. 7 for the unimodal quadratic map $x'=R(1-2x^2)$ at the critical parameter value

$R = R_c$. As with the example discussed in Section 3.1, the range of α values that characterizes the singularity of the corresponding invariant measure is determined by the most rarefied and the most concentrated interval in the set. It has been shown in Refs. 30–32 that the largest interval of the 2^n-cycle scales like $\alpha_{\mathrm{PD}}^{-n}$, while the smallest one scales like $\alpha_{\mathrm{PD}}^{-2n}$, where $\alpha_{\mathrm{PD}} = 2.5029...$ is the universal scaling factor involved in the renormalization operation. Since the measure of each interval is simply $\mu_n = 2^{-n}$, we get from the definition (2.21) of the dynamical free energy and the properties of the Legendre transform that the support of $G^D_{z=2}(\alpha)$ is defined by

$$\alpha \in [\log \alpha_{\mathrm{PD}}/\log 2, \log \alpha^2_{\mathrm{PD}}/\log 2] \tag{4.1}$$

Now, if again one uses the relation (2.25) between the dynamical and uniform Gibbs free energy, one gets for the range of α values for $G^U_{z=2}(\alpha)$

$$\alpha \in [\log 2/(2 \log \alpha_{\mathrm{PD}}), \log 2/\log \alpha_{\mathrm{PD}}] \tag{4.2}$$

Using the renormalization group prediction for α_{PD}, one predicts $\alpha \in [0.3777..., 0.7555...]$ which is in good agreement with the numerical calculation of $G^U_{z=2}(\alpha)$. As emphasized by Halsey *et al.*,[7] the existence of a quadratic fixed point for the renormalization group is reflected not only in (4.2) but in the universality of the whole spectrum of singularities $G^U_{z=2}(\alpha)$, which does not depend on the specific shape of the quadratic map one uses to generate the Cantor set.

Again this result can be checked by investigating the homeomorphism that conjugates two critical quadratic maps as in Section 3.3. The result obtained when considering the map given in Ref. 7 and the logistic map $x' = Rx(1-x)$ confirms the universality conjecture: the homeomorphic conjugacy is found to scale with the same exponent $\alpha = 1$ at each point of the Cantor set. The smoothness of the homeomorphic conjugacy extends to the case of unimodal maps that are not quadratic but have the same type of extremum. This is not surprising with respect to the renormalization group analysis, since it is well established that the order z of the local maximum determines universality classes.[23,30–32] Moreover, the computation of the Gibbs free energy $G_{z,z'}(\alpha)$ for $z \neq z'$ strongly suggests that the scaling of the homeomorphism that conjugates two maps from different universality classes is no longer trivial. A finite range of α values is found to depend on the specific values of z and z' and thus indicates that the associated measure is singular. Upon reproducing the analysis developed in Section 3.4, we obtain a similar prediction:

$$\operatorname{supp} G_{z,z'}(\alpha) = [\min, \max]\{\alpha_{z'}(0)/\alpha_z(0), z'\alpha_{z'}(0)/z\alpha_z(0)\} \tag{4.3}$$

when using the family of critical maps of the interval $[-1, 1]$:

$x' = 1 - R\,|x|^z$ $(z > 1)$. From the renormalization group approach one has $\alpha_z(0) = \log 2/\log \alpha_{\mathrm{PD}}$, and thus we end with the prediction

$$\operatorname{supp} G_{z,z'}(\alpha) = [\min, \max] \left\{ \frac{\log \alpha_{\mathrm{PD}}(z)}{\log \alpha_{\mathrm{PD}}(z')}, \frac{z' \log \alpha_{\mathrm{PD}}(z)}{z \log \alpha_{\mathrm{PD}}(z')} \right\} \tag{4.4}$$

Now if we come back to (4.2), we know from previous work[37] that $\alpha_{\mathrm{PD}}(z)$ is a decreasing function of z, with $\lim_{z \to 1} \alpha_{\mathrm{PD}}(z) = +\infty$ and $\lim_{z \to \infty} \alpha_{\mathrm{PD}}(z) = 1$. This implies that the support of $G_z^U(\alpha)$ is a monotone-increasing function of z. Indeed, according to Collet *et al.*,[23,38] for maps such that $z = 1 + \varepsilon$, it can be shown that $\alpha_{\mathrm{PD}}(1 + \varepsilon) = -\varepsilon \log \varepsilon + O(\varepsilon)$, which yields

$$\lim_{\varepsilon \to 0} \operatorname{supp} G^U_{z=1+\varepsilon}(\alpha) = [\log 2/(2 \log(-\varepsilon \log \varepsilon)), \log 2/\log(-\varepsilon \log \varepsilon)] \tag{4.5}$$

In the opposit limit, the computer-assisted theorem of Eckmann and Wittwer[39] states that $\lim_{z \to \infty} \alpha_{\mathrm{PD}}^{-z}(z) = 0.033381$, or equivalently $\alpha_{\mathrm{PD}}(z) = 1 + 3.40/z + O(1/z^2)$.[40] Consequently, the support of $G_z^U(\alpha)$ increases from the lower limit (4.5) to the upper limit

$$\lim_{z \to \infty} \operatorname{supp} G_z^U(\alpha) = [(z \log 2)/6.80, (z \log 2)/3.40] \tag{4.6}$$

We note that the range of α values is thus found to increase linearly in z at large z.

4.2. The Set of Irrational Winding Numbers at the Onset of Chaos

As already discussed in Section 3.1, universality ideas have also had some success in describing the transition to chaos for diffeomorphisms on the circle. The first example of a universal scaling for critical maps was discovered by studying the golden mean trajectories at the onset of chaos.[7,15-17] The next example is the universality of the Hausdorff dimension $D_0 = 0.87...$ of the set of irrational winding numbers (which is of zero Lebesgue measure) at the onset of chaos.[41,42] The complementary set is the set of mode lockings, which is best understood in terms of the devil's staircase representing the dressed winding number as a function of the bare one [Ω in the sine map (3.1)]. This global statement has been recently questioned, since "the thermodynamics is different from that for the fractal sets which scale everywhere geometrically, such as the period-doubling attractor."[19] The dynamical free energy $F^D(\beta)$ has been shown to exhibit a phase transition between (i) a monotonically decreasing behavior at large and negative β values, where the thinnest intervals located around the golden mean $(W_n = F_n/F_{n+1})$ are dominating and shrink exponentially with a universal scale factor $l_{\min} \sim \delta^{-n}$ $[\delta(z=3)=2.833612...]$[15-17]; and (ii) a logarithmic convergence (when going further and further into the Farey sequence) to $F^D(\beta)=0$ at large and positive β values, where the fatest intervals corresponding to the harmonic sequence $1/q \to 0$ are dominating and shrinks with a power law[41,43,44]: $l_{\max} \sim q^{-3}$. The Hausdorff dimension is actually determined by the transition from geometric scalings that we know are universal[15-17] to harmonic scalings that we know are not.[44] Unfortunately, the application of a numerical convergence acceleration algorithm to a variety of cubic circle maps does not provide unambiguous evidence for or against the universality of the Hausdorff dimension of the set of irrational winding numbers.[19]

The nonuniversality of the whole spectrum of fractal dimensions of this strange set should be easier to verify numerically. Both the dynamical and uniform Gibbs free energy have been computed using the sine map (3.1) in Refs. 19 and 7, respectively. There exists to our knowledge no equivalent computation with different critical cubic maps. From the previous discussion it seems that the nonuniversality of the harmonic sequence should be manifested more clearly on the increasing branch of the dynamical Gibbs free energy (respectively decreasing branch of the uniform Gibbs free energy), which is governed by the most rarefied region of the fractal set. However, we do not really know how important these non-universal effects are or whether they are accessible numerically with the convergence acceleration algorithm. In spite of these technical difficulties, it is very likely that the formalism defined in Section 2 applies to this fractal set, and that the computation of the homeomorphism that maps the sets of irrational winding numbers generated by two different cubic maps should in principle exhibit a nontrivial spectrum of singularities.

We have shown in this paper now homeomorphic conjugacies may change the local scaling behavior of an invariant measure of a one-dimensional dynamical system. A straightforward generalization of Section 2 shows that these results extend to higher dimensions at least for those systems that show uniform scaling behavior in the following sense. Let μ be a probability measure on $[0, 1]^n$; this measure shows exact scaling if

$$\lim_{\mathbf{x} \in U, L(U) \to 0} \log \mu(U)/\log L(U) = \alpha(\mathbf{x}) \tag{4.7}$$

where U is any neighborhood of $\mathbf{x}$ in $[0, 1]^n$.

To illustrate this result, let $\mathbf{y}(\lambda) = \mathbf{y}(0) + \lambda \mathbf{e}$ be the straight line passing through $\mathbf{y}(0)$ with direction $\mathbf{e}$ ($|\mathbf{e}| = 1$). Then the probability measure concentrated on this one-dimensional subspace can be written as $\mu_{(\mathbf{y})} = \mu(\ /\mathbf{y})$

where $\mu(\ /\mathbf{y})$ is a conditional probability. Now if we assume that μ scales exactly, then the scaling exponent $\alpha_{(\mathbf{y})}(\mathbf{x})$ associated to $\mu_{(\mathbf{y})}$ via the relations (1.5) and (1.6) does not depend on the direction $\mathbf{e}$, which means that *the measure μ scales isotropically.*

In general the measure does not satisfy this condition, e.g., for measures that are concentrated on foliated Cantor-like structures, such as the Henon model.[45] Hence, the scaling along the foliations may be trivial, whereas in the other directions the scaling reflects the Cantor-like structure. Thus, we expect the isotropic multiplicative interaction (2.7) between the scaling exponents to be replaced by much more complicated nonisotropic expressions. We hope to elaborate on this point in a forthcoming communication.

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